Math 3235 Probability Theory

1/26/23

Section 2.2

Examples of Discrete r.v.

Bernoulli r.v.

Only Two possible outcomes

$X \in \{0, 1\}$ $\qquad p$

$$p = \mathbb{P}(X = 1)$$

$$q = (1-p) = \mathbb{P}(X = 0)$$

Geometric

Position of first H in a sequence of coin flip.

$X \in \mathbb{N} = \{1, 2, \ldots \infty\}$

$$p_X(x) = \mathbb{P}(X = x) = p(1-p)^{x-1}$$
$$= p\, q^{x-1}$$

$$\sum_{x=1}^{\infty} p(x) = 1$$

Binomial

Repeat a Bernoulli ex. N Times count The number of 1.

My Bernoulli ex have par p prob of getting 1 is p.

In a specific outcome of The N flips I will have x 1 and N-x 0

Prob of each specific outcome is

$$p^x (1-p)^{N-x}$$

The number of outcomes with x 1 and $N-x$ 0 is

$$\binom{N}{x} = \frac{N!}{(N-x)!\, x!}$$

$$p(x) = \mathbb{P}(X = x) = \binom{N}{x} p^x (1-p)^{N-x}$$

Binomial p.m.f. with par N, p.

We need

$$\sum_x p(x) = 1$$

Newton binomial.

$$(a+b)^N = \sum_{n=0}^{N} \binom{N}{n} a^n b^{N-n}$$

$$\underbrace{(a+b)(a+b)\ \dots\ (a+b)}_{N\text{-Times}}$$

$$p(x) = \sum_{x=0}^{N} \binom{N}{x} p^x (1-p)^{N-x} =$$

$$= (p + (1-p))^N = 1$$

$\binom{N}{x}$ binomial coefficients

$$\binom{N}{x} = \binom{N}{N-x}$$

Total population N

and n_1 are Taller Than 6'

$n_2 = N - n_1$ That are shorter

If I extract a person at random.

X resot of my question

1 Taller

0 not

X_1 is bernoulli $p = \frac{n_1}{N}$

X_2 is bernoulli $p = \frac{n_1}{N}$

ind. of X_1

$\vdots$

X_{10} is bernoulli $p = \frac{n_1}{N}$ ind.

$Y = \sum_{i=1}^{10} X_i$ number of Tall people in my sample

Y is binomial $10, \; p = \frac{n_1}{N}$

Hypergeometric

N n_1 S

n_2 F

I select m individual from The Total population of N

How many S will I have among The m selected ones?

X

$m = 1$

$\mathbb{P}(X = 0) = \frac{n_2}{N}$ $\mathbb{P}(X = 1) = \frac{n_1}{N} = p$

$m = 2$

$$\mathbb{P}(X=0) = \frac{n_2}{N}\frac{(n_2-1)}{N-1}$$

$$\mathbb{P}(X=2) = \frac{n_1}{N}\frac{(n_1-1)}{N-1}$$

0 1   1 0

$$\mathbb{P}(X=1) = \frac{n_2}{N}\frac{n_1}{N-1} + \frac{n_1}{N}\frac{n_2}{N-1} =$$

$$= 2\frac{n_1 n_2}{N(N-1)}$$

N m

$$\mathbb{P}(X=x) = \frac{\binom{n_1}{x}\binom{n_2}{m-x}}{\binom{N}{m}}$$

Hypergeometric N, n_1, m

$(n_2 = N - n_1)$

Poisson distribution.

X is poisson with par λ

$$p(x) = \mathbb{P}(X = x) = \frac{1}{x!}\lambda^{x} e^{-\lambda} \quad \text{with } x = 0, 1, \dots$$

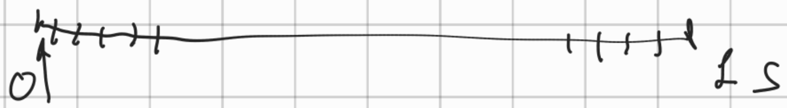
$$\sum_{x=0}^{\infty} \frac{\lambda^{x}}{x!} = e^{\lambda}$$

$$\sum_{x=0}^{\infty} p(x) = 1$$

My server can handle 1000 req in a sec.

Prob (actual number is > 1000)

Poisson process.



N very large subinterval

call X_i the number of arrival in the i-th interval.

If The intervals are short enough:

1) The prob. of 2 arrivals in The same interval is 0. $\left(O\left(\frac{1}{N^2}\right)\right)$

2) The prob of 1 arrival is $\frac{\lambda}{N}$ for some par λ.

3) The numbers of arrivals in The intervals are independent!

Y is The number of arrival is 1 second

Y binomial N $\frac{\lambda}{N}$

$$\mathbb{P}(Y=y) = \binom{N}{y}\left(\frac{\lambda}{N}\right)^y\left(1-\frac{\lambda}{N}\right)^{N-y} =$$

$$= \lambda^y \frac{N!}{(N-y)!\,N^y} \frac{1}{y!} \left(1 - \frac{\lambda}{N}\right)^{N-y}$$

$$\frac{N!}{(N-y)!\,N^y} = \frac{N(N-1)\ldots(N-y+1)}{N^y} =$$

$$= 1\left(1-\frac{1}{N}\right)\left(1-\frac{2}{N}\right)\ldots\left(1-\frac{y+1}{N}\right)$$

$$\lim_{N\to\infty} \frac{N!}{(N-y)!\,N^y} = 1$$

$$\left(1-\frac{\lambda}{N}\right)^{N-y} = \left[\left(1-\frac{\lambda}{N}\right)^{N}\right]^{1-\frac{y}{N}} \quad 1-\frac{y}{N} \to 1$$

$$\lim_{N\to\infty} \left(1-\frac{\lambda}{N}\right)^{N} = e^{-\lambda}$$

$$\lim_{N\to\infty} \left(1-\frac{\lambda}{N}\right)^{N-y} = e^{-\lambda}$$

$$\lim_{N \to \infty} \mathbb{P}(Y = y) = \frac{\lambda^y}{y!} e^{-\lambda}$$

Poisson!